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Reexamining the determination of the fractal dimension of river networks

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Abstract. In analyzing the literature on the fractal nature of river networks one can recognize several points that require clarification or additional investigations. For instance, many interpretations, with different conclusions, have been given to the important empirical results found in the previous literature on river geomorphology. Similarly, the argument concerning a possible "more likely" value of the fractal dimensions of river networks has not yet found a convincing answer. In an attempt to shed more light on the topic, particularly on the latter point, we have reexamined the basic aspects related to self-similarity in river networks using uniquely the fundamental concepts of fractal geometry. This approach leads to relations for the determination of the fractal dimensions which hold exactly for deterministic self-similar trees and are compatible with related findings reported in literature. Results obtained from application of these relations to several river networks in southern Italy indicate very low variability around the average values of about 1.7, 1.5, and 1.1 for the fractal dimensions of the whole river network, of its topological equivalent, and of its individual streams, respectively. This outcome, in addition to comparable results taken from the literature, provides new arguments to the hypothesis that natural networks tend to have the same fractal dimensions.

1. Introduction

The use of the concepts of fractal geometry for the analysis of river networks has proven, in the recent past, to lead to important results on the interpretation of scaling properties concerning several morphological indices of the basins and of similarities recognized between apparently different networks. River networks have been recognized to be fractal structures, presenting self-similar properties over a significant range of scales. The impact of these results is significant in studies on basin evolution and on determination of the channel network response to rainstorms.

Mandelbrot [1982] first hypothesized the fractal nature of rivers and introduced fractal objects similar to river networks. Much work has since been done in exploring fractal properties of rivers and of drainage networks and in exploring relationships between fractal dimensions of different structures. Published literature related to this field may be divided into two groups of papers. A first group of papers [e.g., *Tarboton et al.*, 1988, 1990; *La Barbera and Rosso*, 1989, 1990; *Marani et al.*, 1991; *Beer and Borgas*, 1993] deals with investigations on the fractal behavior of river networks on the basis of *Horton*'s [1945] and *Strahler*'s [1952] laws, regarded as scaling laws. In the second group of papers [e.g., *Liu*, 1992; *Rigon et al.*, 1993; *Rinaldo et al.*, 1993; *Nikora and Sapozhnikov*, 1993; *Nikora et al.*, 1993; *Peckham*, 1995] fractal properties of drainage networks are reproduced by means of simulation models. An approach that can be considered aside is the one documented by *Fiorentino and Claps* [1992b], *Claps and Fiorentino* [1993], and *Claps and Oliveto* [1994], in which the informational entropy was used to derive a number of fractal properties of networks based on comparisons with purely geometric fractal trees.

Most of the results reported in the cited literature were obtained assuming that river network systems are self-similar, that is, they exhibit their scaling properties over a undefined range of scales. On the other hand, more recent literature on river networks [*Snow*, 1989; *Nikora et al.*, 1989, 1993; *Ijjasz-Vasquez et al.*, 1994] has shown that individual river channels should be considered self-affine objects, presenting self-similar behavior only in a defined range of scales. Other work in the literature focuses on the multifractal aspects of the configuration of river networks [e.g., *Marani et al.*, 1994]. Nevertheless, in the hydrologic context river networks are analyzed essentially to find rules and patterns common to different basins to use in models for the estimation of the basin hydrologic response. According to this ultimate goal, the "simplified" framework of self-similar fractal systems has not yet been utilized completely, and we believe that it has still a considerable potential that is worth exploring.

In one of the few approaches in which this framework has been employed, *Claps et al.* [1996] obtained an expression for the most probable hydrological response of self-similar fractal trees and evaluated some criteria of maximum efficiency to identify a possible most probable fractal dimension. In this case, the value of the fractal dimension assumed a particular importance. From the viewpoint of the above approach, as well as from other viewpoints, it is evident that the evaluation of fractal dimensions on river networks needs to be undertaken with particular care.

The purpose of this paper is to try to unify the approaches to the estimation of fractal dimensions in the hypothesis of self-similar behavior of the river network structures. This is done providing a reference framework of deterministic self-similar fractal trees with properties easy to relate to natural networks and sufficient to reproduce some empirical proofs of their

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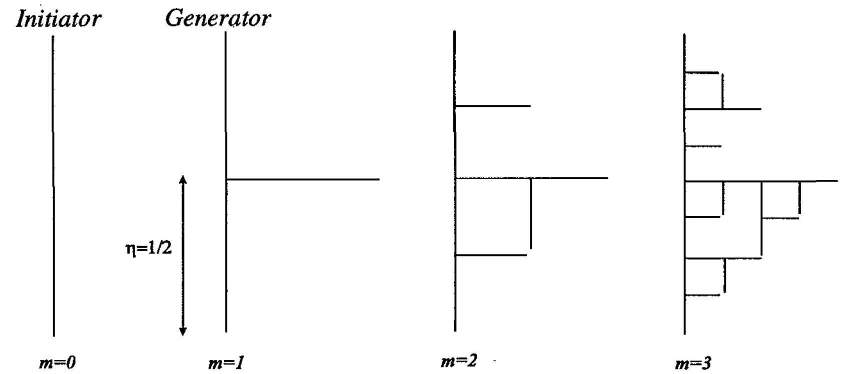


Figure 1. Construction of a self-similar tree; m is the generation index and the generator is the structure with $m = 1$. Number of links and partition of the initiator are $M = 3$, $\eta = 1/2$. Fractal dimension is $\mathfrak{D} = -\ln M/\ln \eta = 1.585$.

fractal nature. In perspective, this approach can even provide new elements useful in the analysis of self-affinity of the natural structures, through the investigation of the departures of the empirical indices from the rules that characterize the properties of deterministic self-similar trees.

In the next section the reference framework of fractal river networks as well as the features and properties of such treelike fractal models are introduced, with particular reference to the combination of the self-similar behaviors in terms of branching and sinuosity.

2. Self-Similar Deterministic Fractal Trees: Structure, Properties, and Composition Rules

The scaling properties of an object can be characterized by a parameter which accounts for how the euclidean measure of the object changes after a change of scale. This parameter is the fractal dimension [*Mandelbrot*, 1982] that represents the dimension of the space in which the measure is scale-invariant.

In a river network, after a change of scale, the global euclidean length changes for two mechanisms. One depends on the increase in the length of individual streams that reveal their paths with greater detail. The other depends on the appearance of new small streams that were previously unobserved, owing to insufficient resolution. Consequently, fractality of river networks can be analyzed from two points of view: one accounts for the sinuosity of rivers and is characterized by the "sinuous" fractal dimension D_s; the other, called the "topological" fractal dimension D_t, reflects the branching characteristics of the network. The combination of the two previous mechanisms produces an object with its own fractal dimension, which is the dimension D of the whole river network.

In principle, the dimension D should be related to the other two dimensions according to a "composition rule" which exemplifies the connection between the two mechanisms of growth. The definition of a framework of deterministic fractal trees, as presented hereafter, allows one to practically build some treelike and sinuous objects with the required properties and to derive their rules of connection.

2.1. Definition and Structure of Self-Similar Fractal Trees

The self-similar trees used in this paper are obtained by a "recursive replacement" algorithm [e.g., *Feder*, 1988, p. 16], in which an initiator (usually a unit-length segment) is replaced by a generator, which is a treelike curve composed by M segments (links) of equal length η. After a replacement, each segment of the generator becomes an initiator and is substituted again (see Figure 1), in a recursive way. If the initiator has unit length, η also has the meaning of a partition ratio, so that $1/\eta = \Delta$, with Δ being the topological diameter of the generator (number of links forming the longest path in the tree). The number of repeated replacements defines the growth stage of this "prefractal" object (strictly speaking, the fractal set is obtained after infinite replacements).

This recursive algorithm produces an object with fractal dimension $\mathfrak{D}$ defined as [*Feder*, 1988]

$$\mathfrak{D} = -\frac{\ln M}{\ln \eta} \tag{1}$$

The tree depicted in Figure 1 has straight links connected with right angles. In self-similar trees like this the main stream of a generic subbasin does not change its length after the recursive replacements. These trees can be used to reproduce the topological structure of river networks and reproduce the growth of their total length (with the change of scale) owing to only the branching mechanism.

To build self-similar trees with self-similar (nonstraight) channels, as those found in nature, the branching structure must be combined with fractal objects aiming to reproduce the sinuosity properties of individual river channels (see, e.g., Figure 2). With reference to these latter, relation (1) applies without modifications, because the definitions of M and η for the sinuous objects are the same as stated above. The fractal

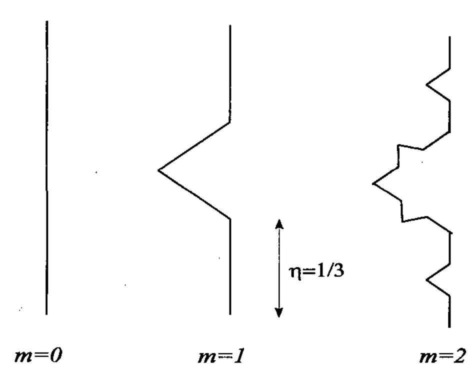
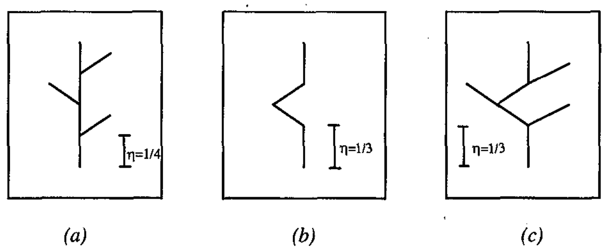


Figure 2. Construction of a fractal "sinuous" curve; $M = 4$, $\eta = 1/3$. Fractal dimension is $\mathfrak{D} = -\ln M/\ln \eta = 1.262$.



M_t=7; η_t=1/4; D_t=1.404 M_s=4; η_s=1/3; D_s=1.262 M=7; η=1/3; D=1.771

Figure 3. Fractal model by *Nikora and Sapozhnikov* [1993]. Subscript t indicates that the generator (Figure 3a) simulates the network topology; subscript s indicates that the generator (Figure 3b) simulates the sinuosity of individual rivers. According to the definition (1), fractal dimension of the compound structure (Figure 3c) is $D = -\ln M/\ln \eta = \ln 7/\ln 3 = \ln 7/\ln 4 \cdot \ln 4/\ln 3 = D_t \cdot D_s$.

curve shown in Figure 2 is of the family of Koch curves [*Mandelbrot*, 1982].

Among the procedures proposed in the literature for composition of branching and sinuosity, discussed in Appendix 1, only a model proposed by *Nikora and Sapozhnikov* [1993], with some additional specifications, is compatible with a framework of deterministic fractals obtained by recursive replacement. *Nikora and Sapozhnikov* [1993, Figure 1] presented a network fractal model in which the generator is obtained combining a Koch curve and a treelike curve, as shown in Figure 3. As can be recognized from the figure, the sinuosity structure in Figure 3b substitutes the mainstream in the geometric tree determining a new length $\eta = 1/3$ for all of the segments of the tree in Figure 3c. As a consequence of this way of obtaining the combined structure in Figure 3c, its fractal dimension is equal to the product of the fractal dimensions of the two initial structures in Figures 3a and 3b:

$$D = D_s D_t \tag{2}$$

As shown in Appendix 1, for unconditional validity of this rule with reference to deterministic self-similar trees the constraint (A3) is needed. This constraint, which is implicitly assumed in the treelike models used in this paper, minimizes the possible alternatives in the construction of the compound structure but does not affect the possibility of building models of river networks with whatever fractal dimension ($1 \leq D \leq 2$). In Appendix 2 it is shown that relation (2) is also asymptotically compatible with the hortonian ordering scheme.

2.2. Basic Properties of Deterministic Fractal Trees

Being fractal objects, recursive replacement trees obey the rule [*Mandelbrot*, 1982, pp. 30, 36]

$$\mathcal{Z} = M_\eta \eta^D \tag{3}$$

where $\mathcal{Z}$ is the fractal measure of the whole tree, (i.e., a length in the space of dimension D independent of the measurement scale), M_η is the number of rulers needed to cover the tree, η is the ruler length, and D is the fractal dimension. The definition of the fractal dimension D of the tree as a whole accounts both for branching and sinuosity. In an analogous way, the fractal measure $\mathcal{L}$ of the main stream is defined as

$$\mathcal{L} = N_\eta \eta^{D_s} \tag{4}$$

with an intuitive meaning of the symbols.

If the length L_0 of the initiator is 1, the fractal measure of the self-similar set is also 1. Based on inductive arguments, if $L_0 \neq 1$ the following relations hold [see, e.g., *Nikora*, 1991, relation 3].

$$\mathcal{Z} = L_0^D \tag{5a}$$

$$\mathcal{L} = L_0^{D_s} \tag{5b}$$

For the class of fractal trees considered here it is particularly convenient to consider the case in which the link length equals η. This condition does not affect structural relations such as (3) and allows one to set up some properties connected to the branching characteristics of the trees. The basic property arising from this assumption attains their topological structure and is a relation between the total number M of links and the topological diameter Δ:

$$M = \Delta^{D_t} \tag{6}$$

This relation can be obtained considering that if Δ_1 and M_1 are the quantities related to the topological generator structure, then after m recursive replacements one obtains $\Delta_m = \Delta_1^m$ and $M_m = M_1^m$. By the elimination of m between these two relations and considering that $\eta_m = 1/\Delta_m$ (for unit-length initiator), using (1) one obtains (6). According to the notation just used, relation (6) should be written as

$$M_m = \Delta_m^{D_t} \tag{7}$$

which underlines the fact that this result holds exactly for complete prefractals. In different contexts, relation (6) was also derived by *Fiorentino and Claps* [1992a, equation 38] through relations between the Horton ratios and the informational entropy of the network, by *Agnese et al.* [1993] as a property of single-scaling infinite topologically random channel networks, and by *Peckham* [1995] in a revised hortonian framework.

With regard to compound fractal trees, if η is included as the link length in relation (6), one obtains $M\eta = (\Delta\eta)^{D_t}\eta^{1-D_t}$ and therefore

$$Z = L^{D_t}\eta^{1-D_t} \tag{8}$$

with Z and L as the euclidean lengths of the whole network and of the main stream, respectively.

Application of the concepts expounded above to natural networks leads to relations adopted for the estimation of the different fractal dimensions, as shown in the next section.

3. Estimation of Fractal Dimensions of River Networks

3.1. Topological Fractal Dimension

The first investigations regarding scaling properties of river networks were focused on the fractal dimension related to the branching [*Tarboton et al.*, 1988, 1990; *La Barbera and Rosso*, 1989, 1990]. Since then, this issue has received very little attention, probably because the validation of the estimates of D_t could not take full advantage of the previous literature on fluvial geomorphology [e.g., *Hack*, 1957; *Gray*, 1961; *Leopold et al.*, 1964; *Gregory and Walling*, 1973]. As a matter of fact, relation

$$D = \frac{\ln (R_B)}{\ln (R_L)} \tag{9}$$

by *La Barbera and Rosso* [1989], insofar as it refers explicitly to the branching after the clarification by *Tarboton et al.* [1990], has represented for long time the only way to estimate D_t.

Only recently, *Peckham* [1995], revising the hortonian approach based on the analysis of Tokunaga's tree graphs, expressed D_t as

$$D_t = \frac{\ln (R_B)}{\ln (R_C)} \tag{10}$$

where R_C is a "number-of-links ratio" certainly more meaningfully connected to the branching mechanism of networks than R_L. This author has brought into the hortonian framework the notion that estimation of D_t must be related uniquely to the topology of the networks (no role of stream lengths). Moving from different premises, *Claps and Fiorentino* [1993] and *Agnese et al.* [1993] had expressed the same concept.

In a series of papers [*Fiorentino and Claps* 1992b; *Claps and Fiorentino*, 1993] the branching properties of river networks were investigated using the informational entropy of river networks, defined [*Fiorentino and Claps*, 1992a; *Fiorentino et al.*, 1993] with respect to their topological structure. New proofs of the self-similar character of river networks with respect to branching and a new method for evaluation of the branching fractal dimension were derived [*Claps and Fiorentino*, 1993] by comparison of some features displayed by the network entropy in deterministic fractal trees and in natural networks.

In particular, in defining the informational entropy of river networks as $S = -\sum_{\delta=1}^{\Delta} p_\delta \ln p_\delta$, with p_δ as the relative number of links at topological distance δ from the outlet, if S_Ω is the entropy of a tree of Horton order Ω and S_0 is the entropy of the generator of a fractal tree, *Claps and Fiorentino* [1993] showed that the relation $S_\Omega = (\Omega - 1)S_0$ holds theoretically for fractal trees and matches exactly an empirical relation between S_Ω and Ω found to hold [e.g., *Fiorentino et al.*, 1993, Figure 8] in natural networks. The same striking analogy was found with respect to the dependence of entropy on the topological diameter. Consequently, the coefficient equivalent to S_0 estimated on natural networks was used to evaluate the branching fractal dimension through an empirical equation relating S_0 and D_t for a set of deterministic fractal trees. The trees adopted have the same characteristics of the one shown in Figure 1.

Application of this method to eight rivers in southern Italy showed a very small variability (± 0.029) of the estimated D_t around a mean value of 1.5. The same mean value arose with the classical estimation of D_t made through (9), even though with a much larger variability (± 0.169).

A problem that arises in using the Horton-Strahler framework for the estimate of D_t concerns the determination of the ratios R_B and R_L (or R_C) because consideration of the whole basin as a "mature" watershed of order Ω is often misleading. What often happens is that the basin outlet is little downstream of a junction of two subbasins of order $\Omega - 1$. In these cases the basin is not really representative of a fully developed network of order Ω, and this affects the estimates of Horton ratios. Moreover, *Beer and Borgas* [1993] and *Claps and Oliveto* [1994] showed that application of (9) to pure hortonian networks and to deterministic fractal trees, respectively, provides reliable estimations of the theoretical dimension only asymptotically, that is, far enough from the average configurations of the natural basins analyzed in literature, which seldom have Horton orders greater than 5.

Given all this, to make the determination of D_t completely independent of the Horton ratios and to verify results obtained by *Claps and Fiorentino* [1993], it is proposed here to use relation (6) for estimation of D_t in the form

$$D_t = E\left[\frac{\ln M}{\ln \Delta}\right] \tag{11}$$

The principle, which will be also applied to estimation of the other fractal dimensions, is to fix the ruler (as the average link length) and the scale and to compute the dimension by averaging the values of the ratio $\ln M/\ln \Delta$ obtained for all of the subbasins in the network. If one has a simple method to isolate all subbasins, intended as the basins draining into every link within the network, this technique is simple to automate and does not leave room for subjective choices (e.g., the range of rulers). This method gives a more complete picture of D_t with respect to the relation $D_t = \ln M/\ln \Delta$, proposed by *Agnese et al.* [1993], which is relative to parameters M and Δ of the whole basin only. Differences in the results of estimation are discussed below.

Estimation of D_t was made on 23 river basins in southern Italy (see Figure 4 and Table 1 for their main characteristics). Some of them are subbasins and are dealt with independently, owing to the presence of a runoff gauging station at the basin outlet. It was judged useful to determine results of the analysis for subbasins of a larger basin, as they constitute fixed points of an analysis performed throughout all possible subbasins. The channel networks were digitized as midchannel traces of streams (blue lines) from 1:25000 scale topographic maps, at data point spacing of approximately 25 m true scale, or 1 mm map distance.

To verify also the dispersion of values of D_t within the network, the dimension was computed by regression between the couples of Δ and M available for all subbasins. Results obtained in the estimation of D_t are reported in Table 2, while Figure 5 shows an example of the regression between Δ and M performed on one of the basins considered. For the basin to which Figure 5 is related, over 6000 subnetworks were considered, that is, every link's subnetwork except for first-order links.

As emerges from the results in Table 2, the average value estimated for D_t is very close to 1.5, with very low standard deviation. In Table 2 are also reported the R^2 of regression and the estimate of the intercept, which is very close to 1 in all of the basins, ensuring that $M = 1$ when $\Delta = 1$. Given the value 1 of the intercept, the estimates of D_t obtained through (11) are practically the same as those reported in Table 2 (the average value over all of the subbasins is 1.508 ± 0.021).

The estimates obtained for D_t confirm the outcome by *Claps and Fiorentino* [1993] and compare well with average estimates obtained by other authors who used relation (9). For instance, *Nikora et al.* [1989] obtained $\bar{D}_t = 1.50 \pm 0.27$ while *Liu* [1992] obtained $\bar{D}_t = 1.55 \pm 0.28$, with standard deviations comparable with the one obtained using (9) on the basins considered here (Table 2). The range of variation of the mean between these analyses is compatible with the high values attained by the standard deviation. Significantly different estimates, particularly if compared to the standard deviation, emerge by using the method proposed by *Agnese et al.* [1993].

Another meaningful comparison can be established with the results obtained by *Moussa and Bocquillon* [1993]. These authors estimate the branching fractal dimension on data extracted by a digital elevation model (DEM) by plotting the log of the sources arising with changes in the support area A versus

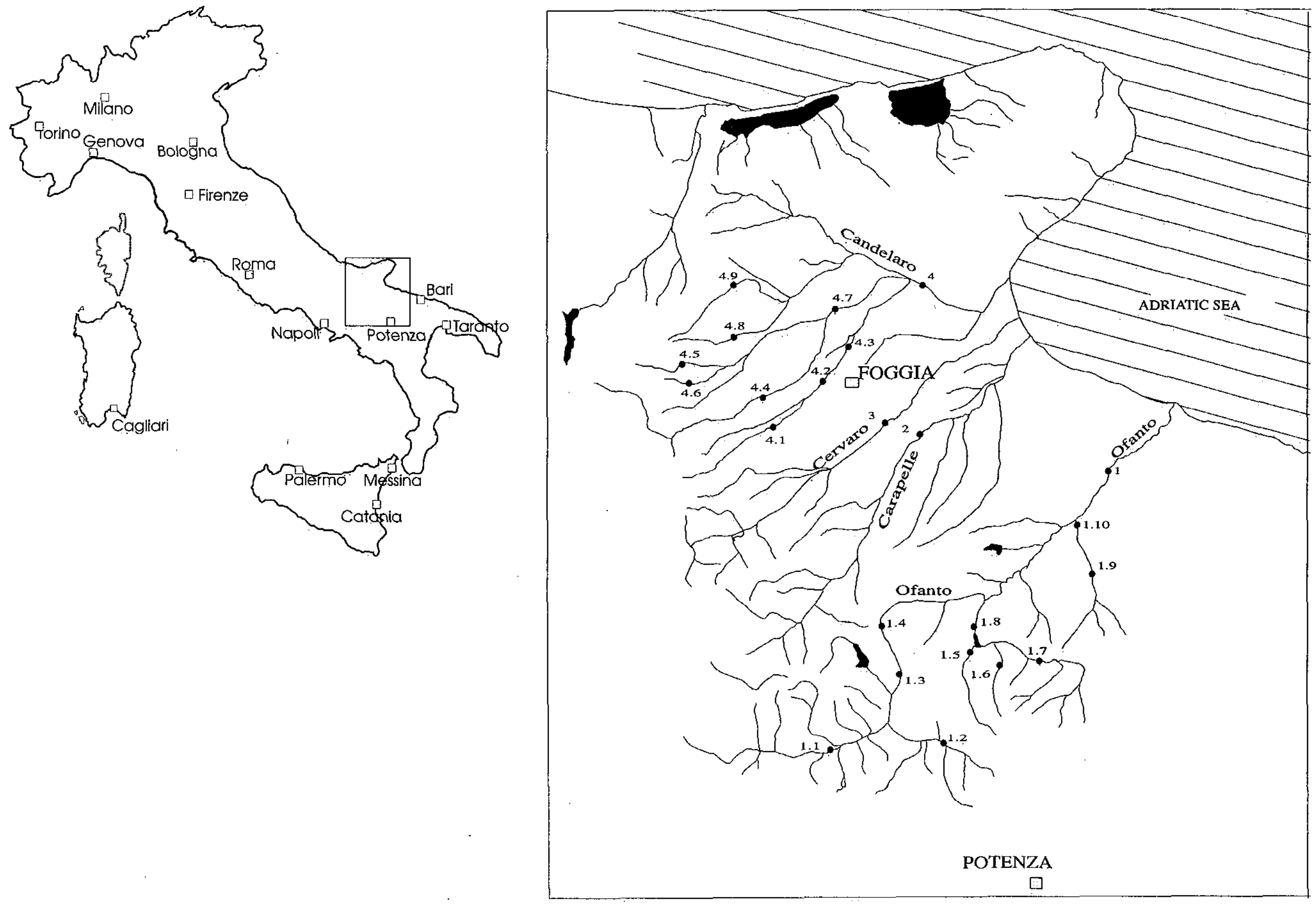


Figure 4. Map of the basins considered in southern Italy.

the log of A/A_0, A_0 being the basin area. This method is somewhat consistent with the one proposed here and produced values of D_t of 1.54, 1.54, and 1.42 for the three basins considered.

In conclusion, application of four classes of methods for estimation of D_t (the one presented here, the one by *Claps and Fiorentino* [1993], the classical one used by *Nikora et al.* [1989] and by *Liu* [1992], and the one used by *Moussa and Bocquillon* [1993]), applied either to digitized data or to DEM-extracted networks, produced results compatible with an average value of $D_t \approx 1.5$. These findings indicate that this value could represent a property of all networks, perhaps as a result of criteria of most efficient configuration.

3.2. Fractal Dimension of River Networks

A combination of the two definitions (3) and (5a) of the fractal dimension D of the network can be used for its actual estimation. Eliminating $\mathscr{L}$ between the two cited relations one obtains

$$D = \frac{\ln M}{\ln (L_0/\eta)} \tag{12}$$

in which L_0 represents the straight-line distance between the two endpoints of the subnetwork mainstream (minimum-path length). Based on the assumption that $\eta = Z/M$, the "characteristic" value of D for a river basin can actually be estimated as the average value within the network:

$$D = E\left[\frac{\ln M}{\ln (L_0/\eta)}\right] \tag{13}$$

Application of (13) to the set of basins under study led to the results shown in Table 3, while Figure 6, relative to one of the basins considered, shows the dispersion associated with the estimation of D. Few negative values obtained for the quantity in parentheses were not considered to compute the mean within the basin.

The following comments arise from observation of the results obtained: (1) For the set of networks considered, the average value obtained for D is about 1.7; (2) if one gives confidence to the estimate obtained through (13), the estimate obtained using Z, L_0, and M of the whole network (relation (12)) is again a little biased, even though the difference is less than one standard deviation; and (3) it is also worth adding that *Moussa and Bocquillon* [1993] estimated a value $D = 1.74$ on the first of the three basins they analyzed (the results on the others were not reported). This result, obtained from DTM-extracted data, contrasts with the values obtained by *Tarboton et al.* [1988] through the box-counting method that led them [*Tarboton et al.*, 1990] to state that D should tend to 2.

The ensemble of our and *Moussa and Bocquillon*'s [1993] findings allow us to comment that (1) a value of $D \approx 1.7$ of the fractal dimension D of river networks could be a property of all networks, even though the elements supporting this conjecture are less than those available with respect to the dimension D_t;

Table 1. Main Characteristics of the Southern Italy River Networks Analyzed

Code	Drainage Basin	Area, km^2	Z, km	L, km	M	Δ	Ω	R_B	R_L
1.0	Ofanto a S.Samuele	2702.8	5841.9	152.7	13385	264	8	3.69	2.18
1.1	Ofanto a Cairano	266.4	840.2	37.1	2383	94	6	4.00	2.14
1.3	Ofanto a Monteverde	1017.5	2755.7	62.4	6627	145	7	4.06	2.25
1.4	Ofanto a Rocchetta	1111.0	2995.3	77.5	7143	189	7	4.13	2.25
1.2	Atella	175.9	510.6	21.3	1165	80	6	3.78	2.18
1.5	Arcidiaconata	123.9	277.7	23.4	507	50	5	4.12	2.39
1.6	Lapilloso	28.5	66.7	12.0	143	36	4	4.34	2.28
1.7	Venosa a p.te ferroviario	204.0	496.5	29.5	1089	86	5	4.78	2.62
1.8	Venosa a p.te S.Angelo	263.0	640.3	40.4	1415	109	5	5.09	2.87
1.9	Locone a p.te Brandi	219.4	566.2	30.6	1577	110	7	3.10	1.81
1.10	Locone a p.te Canosa-Lavello	278.6	668.2	41.4	1789	132	7	3.19	1.81
2.0	Carapelle	714.9	1134.5	85.2	2149	105	6	3.94	2.39
3.0	Cervaro	539.3	1022.6	87.5	1837	131	6	4.14	2.48
4.0	Candelaro a p.te 13 luci	1777.9	2748.5	72.3	5029	123	7	3.72	2.25
4.1	Celone a p.te S.Vincenzo	92.5	191.5	28.3	361	42	5	3.83	2.73
4.2	Celone a p.te Foggia-Lucera	222.2	355.1	46.3	577	50	5	4.10	2.74
4.3	Celone a p.te Foggia-S.Severo	233.5	361.8	49.9	583	53	5	4.09	2.75
4.4	Vulgano	94.1	195.4	24.9	385	37	5	3.79	2.26
4.5	Salsola a Casanova	44.1	99.1	14.8	199	31	5	3.28	2.30
4.6	Casanova a p.te Lucera-Motta	57.3	126.1	16.8	245	26	5	3.44	2.55
4.7	Salsola a p.te Foggia-S.Severo	455.4	764.3	50.1	1349	59	6	3.63	2.28
4.8	Triolo	55.9	140.7	21.9	381	57	5	3.72	2.31
4.9	Canale S.Maria	58.1	131.0	15.9	317	48	5	2.56	2.57

Z, total length; L, main stream length; M, total number of links; Δ, topological diameter; Ω, Horton order; R_B, stream number ratio; R_L, stream length ratio.

(2) the box-counting method applied to the whole network does not give an estimate of D_t (as assumed by *Tarboton et al.* [1988]), but of D; (3) according to the outcome by *Moussa and Bocquillon* [1993], which provides $D \neq 2$, one could think that the box-counting method could be responsible for often producing $D = 2$ in DEM-based networks [see *La Barbera and Rosso*, 1989]; and (4) with reference to the composition rule (2), the average value of about 1.7 found on our 23 basins is consistent with D_t being about 1.5 and with the literature values reported for $D_s(\approx 1.14)$, because $\bar{D}_s = 1.7/1.5 = 1.13$.

A final comment on the methods formerly proposed in the literature for the estimation of D is in order. Owing to the great deal of existing data reporting relations between main stream length or network total length and basin area and to hypotheses based on the hortonian framework, most of the relations proposed in the past for estimation of D contain area-related parameters, particularly the stream area ratio R_A.

Table 2. Estimation of D_t as Exponent of the Linear Regression Between ln M and ln Δ, According to Relation $M = a\Delta^{D_t}$, and Results of Two Other Estimation Methods

Code	Drainage Basin	a	D_t	R^2	ln R_B/ln R_L	ln M/ln Δ
1.0	Ofanto a S.Samuele	0.909	1.566	0.977	1.675	1.704
1.1	Ofanto a Cairano	0.870	1.608	0.971	1.822	1.712
1.3	Ofanto a Monteverde	0.942	1.544	0.973	1.728	1.768
1.4	Ofanto a Rocchetta	0.910	1.570	0.974	1.749	1.693
1.2	Atella	0.994	1.519	0.977	1.706	1.611
1.5	Arcidiaconata	0.981	1.521	0.984	1.625	1.592
1.6	Lapilloso	1.172	1.380	0.992	1.781	1.385
1.7	Venosa a p.te ferroviario	0.927	1.546	0.968	1.624	1.570
1.8	Venosa a p.te S.Angelo	0.956	1.522	0.974	1.543	1.546
1.9	Locone a p.te Brandi	0.997	1.515	0.989	1.907	1.566
1.10	Locone a p.te Canosa-Lavello	0.990	1.518	0.990	1.955	1.534
2.0	Carapelle	0.958	1.512	0.979	1.574	1.649
3.0	Cervaro	1.026	1.484	0.991	1.564	1.542
4.0	Candelaro a p.te 13 luci	0.901	1.557	0.972	1.620	1.771
4.1	Celone a p.te S.Vincenzo	0.991	1.470	0.981	1.337	1.576
4.2	Celone a p.te Foggia-Lucera	1.024	1.430	0.970	1.395	1.625
4.3	Celone a p.te Foggia-S.Severo	1.046	1.414	0.970	1.397	1.604
4.4	Vulgano	0.909	1.577	0.979	1.634	1.649
4.5	Salsola a Casanova	0.975	1.558	0.983	1.426	1.541
4.6	Casanova a p.te Lucera-Motta	0.999	1.469	0.966	1.320	1.688
4.7	Salsola a p.te Foggia-S.Severo	0.946	1.538	0.976	1.564	1.768
4.8	Triolo	1.045	1.494	0.992	1.591	1.470
4.9	Canale S.Maria	1.080	1.426	0.991	1.392	1.488
Mean		...	1.510	...	1.606	1.611
Standard deviation		...	0.057	...	0.175	0.100

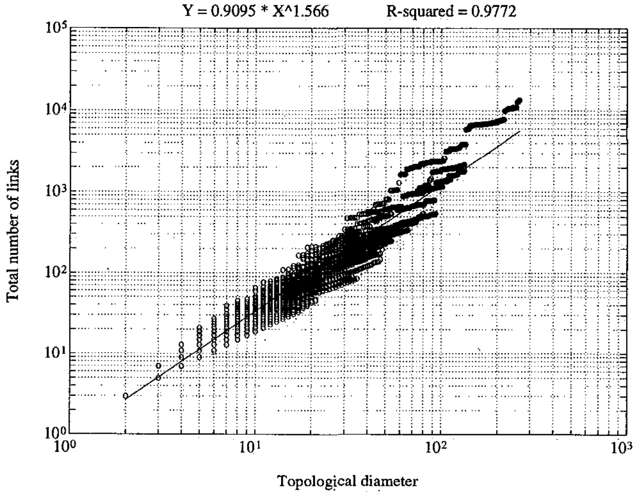


Figure 5. Regression between the total number of links M and the topological diameter Δ for all subbasins, with $\Delta > 1$, of the Ofanto river at S.Samuele. The estimated value of D_t is 1.566.

However, when relations based on R_A are used, results turn out to be unreliable, as can be recognized by the estimates provided by *Rosso et al.* [1991].

In particular, *Rosso et al.* [1991] compared two relations for estimation of D. The first is

$$D = 2 \cdot \frac{\ln (R_B)}{\ln (R_A)} \tag{14}$$

Table 3. Estimates of the Fractal Dimension D of the River Network

Code	Drainage Basin	D*	D†
1.0	Ofanto a S.Samuele	1.746	1.777
1.1	Ofanto a Cairano	1.798	1.887
1.3	Ofanto a Monteverde	1.809	1.931
1.4	Ofanto a Rocchetta	1.802	1.910
1.2	Atella	1.740	2.077
1.5	Arcidiaconata	1.705	1.859
1.6	Lapilloso	1.742	1.673
1.7	Venosa a p.te ferroviario	1.729	1.893
1.8	Venosa a p.te S.Angelo	1.727	1.829
1.9	Locone a p.te Brandi	1.779	1.909
1.10	Locone a p.te Canosa-Lavello	1.769	1.791
2.0	Carapelle	1.778	1.646
3.0	Cervaro	1.737	1.734
4.0	Candelaro a p.te 13 luci	1.837	1.923
4.1	Celone a p.te S.Vincenzo	1.629	1.592
4.2	Celone a p.te Foggia-Lucera	1.642	1.553
4.3	Celone a p.te Foggia-S.Severo	1.643	1.581
4.4	Vulgano	1.639	1.638
4.5	Salsola a Casanova	1.694	1.640
4.6	Casanova a p.te Lucera-Motta	1.630	1.688
4.7	Salsola a p.te Foggia-S.Severo	1.713	1.744
4.8	Triolo	1.606	1.515
4.9	Canale S.Maria	1.607	1.625
Mean		1.717	1.757
Standard deviation		0.070	0.149

*Obtained through (13) averaged on all subnetworks in the basin.
†Obtained using (12) considering the three parameters related to the whole network.

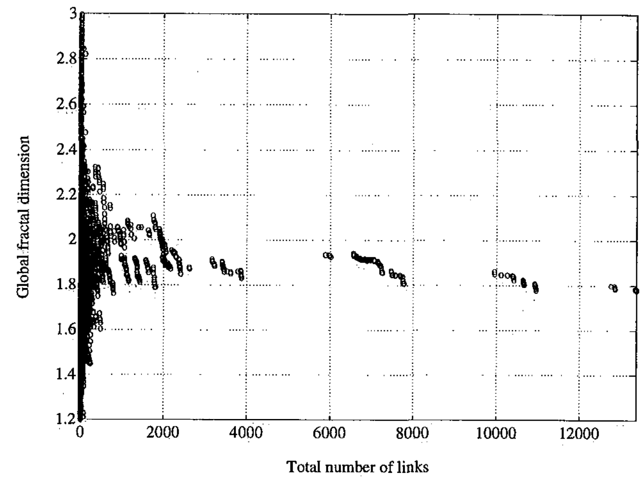


Figure 6. Fractal dimension D of the river network evaluated through (13) for all subbasins of Ofanto river at S.Samuele. Average value of D is 1.746.

which was obtained also by *Nikora* [1988] and *Nikora et al.* [1989]. Relation (14) is obtained combining

$$D_s = 2 \cdot \frac{\ln (R_L)}{\ln (R_A)} \tag{15}$$

with the composition rule $D = D_s \ln R_B/\ln R_L$. With (14), *Rosso et al.* [1991] obtained results in a very wide range (1.44 ÷ 1.91). The second relation arises from the interpretation given by the above authors of the relation

$$Z \propto A^b \tag{16}$$

found in empirical studies based on large samples [e.g., *Gregory and Walling*, 1973, Figure 5.12]. *Rosso et al.* [1991] suggested that the total network length should depend on the square root of the area according to

$$Z \propto (A^{1/2})^D \tag{17}$$

which is an expression equivalent to

$$L^{1/D_s} \propto A^{1/2} \tag{18}$$

proposed by *Mandelbrot* [1982] to relate fractality of individual streams to the empirical relation found by *Hack* [1957] between area and mainstream length. Incidentally, *Mandelbrot* [1982] assumed $Z \propto A$, according to the postulate that networks were plane filling. It is worth remarking that relations (17) and (18) implicitly assume $A^{1/2}$ as equivalent to L_0, so that they recall the definition formula:

$$Z = \eta^{1-D} L_0^D \tag{19}$$

obtained combining (3) and (5a) and substituting M_η with Z/η. Using relation (17), *Rosso et al.* [1991] obtained again a significantly wide range of estimated $D(1.67 \div 1.90)$ even though the value 1.67 has a particular relevance since derives from the average of exponents found on a large sample by *Gregory and Walling* [1973].

On the basis of only these results and even adding the result of the application of (17) to our sample of 23 basins (D = 1.93, with $R^2 = 0.97$) it is very difficult to rate the efficiency of (17) as a method to estimate D, even because the theoretical formula (19) does not correspond exactly to (17) since η^{1-D} is

Table 4. Estimates of the Sinuosity Fractal Dimension D_s

Code	Drainage Basin	D_s*	D_s†	D_s‡	D_s§
1.0	Ofanto a S.Samuele	1.115	1.135	1.124	1.101
1.1	Ofanto a Cairano	1.118	1.174	1.118	1.134
1.3	Ofanto a Monteverde	1.172	1.251	1.125	1.101
1.4	Ofanto a Rocchetta	1.148	1.217	1.126	1.123
1.2	Atella	1.145	1.367	1.100	1.123
1.5	Arcidiaconata	1.121	1.222	1.109	1.115
1.6	Lapilloso	1.262	1.212	1.101	1.086
1.7	Venosa a p.te ferroviario	1.118	1.224	1.109	1.119
1.8	Venosa a p.te S.Angelo	1.135	1.202	1.112	1.126
1.9	Locone a p.te Brandi	1.174	1.260	1.148	1.142
1.10	Locone a p.te Canosa-Lavello	1.165	1.180	1.145	1.121
2.0	Carapelle	1.176	1.089	1.138	1.100
3.0	Cervaro	1.170	1.168	1.161	1.174
4.0	Candelaro a p.te 13 luci	1.180	1.235	1.259	1.056
4.1	Celone a p.te S.Vincenzo	1.108	1.083	1.158	1.080
4.2	Celone a p.te Foggia-Lucera	1.148	1.086	1.156	1.078
4.3	Celone a p.te Foggia-S.Severo	1.162	1.118	1.157	1.083
4.4	Vulgano	1.039	1.039	1.093	1.078
4.5	Salsola a Casanova	1.087	1.053	1.122	1.050
4.6	Casanova a p.te Lucera-Motta	1.110	1.149	1.112	1.075
4.7	Salsoia a p.te Foggia-S.Severo	1.114	1.134	1.155	1.063
4.8	Triolo	1.075	1.014	1.113	1.041
4.9	Canale S.Maria	1.127	1.140	1.106	1.029
Mean		1.138	1.163	1.132	1.096
Standard deviation		0.045	0.083	0.035	0.035

Obtained from D of Table 3 using the composition rule $D = D_t D_s$ with values of D_t taken from Table 2.
†Obtained from D† of Table 3 using the composition rule $D = D_t D_s$ with values of D_t taken from Table 2.
‡Obtained through (22) averaged on all subnetworks in the basin.
§Obtained using (21) considering the three parameters related to the whole network.

also a function of D. All considered, it seems that there are quite few justifications for the use of area-based methods for the estimation of the fractal dimension of river networks.

3.3. Estimation of the Fractal Dimension of Stream Courses

Since the hypothesis of *Mandelbrot* [1982], who related *Hack*'s [1957] law to the possibility that the main stream in a network could be a fractal curve, fractal properties of watercourses have been widely investigated. According to the empirical results by *Hack* [1957], main stream length and basin area are related as

$$L \propto A^{\alpha} \tag{20}$$

with $\alpha \approx 0.6$. Later, *Gray* [1961] estimated on a different set of data a value of $\alpha = 0.568$, which, by means of (18), expressed by *Mandelbrot* [1982], leads to $D_s = 2\alpha = 1.136 \approx 1.14$. This result has been substantially confirmed by direct evaluations of D_s [e.g., *Hjelmfelt*, 1988] obtained generally by means of the Richardson method [e.g., *Feder*, 1988]. A recent review of the most significant papers on this subject can be found in work by *Nikora et al.* [1993].

On the other hand, *Snow* [1989] and *Nikora et al.* [1989, 1993], as well as *Ijjasz-Vasquez et al.* [1994], have worked on a more general classification of the main stream curve as a self-affine, rather then a self-similar, object. In particular, *Nikora et al.* [1989] suggested that relation (18) is not related to self-similarity of river networks, and *Nikora et al.* [1993] provided an explanation of (18) in self-affine terms. At the point reached by the research on this topic, self-similarity of watercourses seems to be limited to relatively small scales only, while the role of other factors, such as valley morphology, cannot be disregarded in a general analysis of river plan forms [*Nikora et al.*, 1989].

Since the frame of analysis adopted here is that of self-similar systems, it is required to verify that tools usually adopted for estimation of the self-similar dimension of individual streams are used with reference to scales compatible with the self-similar behavior. To this end, it must be pointed out that the ruler η used to measure channel lengths, corresponding to the resolution of the digitizing process, is certainly taken in the range of scales defining self-similar behavior. This range, according to *Nikora et al.* [1993], is delimited by the internal and external scales of fractality, respectively related to the river width and the width of the river valley. With this in mind, we have completed the analysis of river networks considering the individual streams as self-similar objects, aiming to clarify some concepts related to this field and to possibly corroborate literature data obtained according to this hypothesis.

In the spirit of the framework of deterministic fractals, the method suggested here for estimation of D_s is to use strictly the definitions (4) and (5b), from which one obtains

$$D_s = \frac{\ln \Delta}{\ln (L_0/\eta)} \tag{21}$$

in which the ruler $\eta = L/\Delta$ represents the average link length. Again, the estimate of D_s for the whole network is suggested as the average of the values computed for all of the network subbasins:

$$D_s = E\left[\frac{\ln \Delta}{\ln (L_0/\eta)}\right] \tag{22}$$

By applying (22) over all the basins in Table 1, we obtained an average value of 1.13 for D_s (Table 4). The dispersion

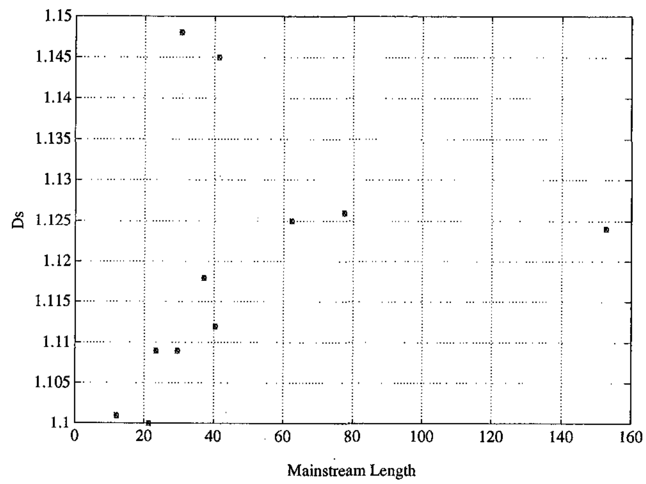


Figure 7. Sinuosity fractal dimension D_s, evaluated through (22), versus main stream length (in kilometers) of the principal subbasins of the Ofanto river at S.Samuele (the subbasins can be identified in Table 1 and Figure 4).

associated with the estimate in each basin is about the same that results from the estimation of D. Figure 7 shows the evolution of D_s among the "main" subbasins within a great basin.

Results obtained in the estimation of D_s can be compared with the results of other methods reported in literature, which consist in direct estimations [e.g., *Hjelmfelt*, 1988] or indirect methods based on the use of the length-area relation (using relation (18)) or of the Horton ratios (using relation (15)). The comparison shows a substantial agreement between our results and these obtained with all of the literature methods, even though it is to emphasize that the value $\bar{D}_s = 1.13$ obtained here is identical to the "reference" value obtained by *Gray* [1961]. It is interesting to consider that our estimation of D_s accounts, through an average value, for the sinuosity of the main streams of all of the subnetworks within each basin.

The similarity of ours and *Gray*'s [1961] results doesn't end here, since we have tested on our basins a little-cited result by *Gray* [1961], who performed a regression between L and L_{ca} (length to the center of area) and obtained $L_{ca} \propto L^{0.96}$, corresponding to $L \propto L_{ca}^{1.04}$. Considering L_{ca} to be analogous to L_0 (both are scale-invariant) the exponent has to be the fractal dimension D_s because arises from the same relation

$$L = \eta^{1-D_s} L_0^{D_s}, \tag{23}$$

equivalent to (19), on which is indirectly based the estimation of $\alpha = D_s/2$. However, the value obtained by Gray is clearly different from 1.14, and exactly the same result (1.06 ± 0.02) arises from averaging the exponents β (not reported here) of the relation $L \propto L_0^{\beta}$ estimated on all the basins of our sample. An additional confirmation comes from estimations made by *Ijjasz-Vasquez et al.* [1993], who obtained 1.05 ± 0.02. The reasons of this discrepancy are not apparent and stimulate further investigations.

With regard to the impact of our results on the composition rule (2), one can notice that the mean value $\bar{D}_s = 1.13$ obtained for the whole sample is identical to the one (Table 4) obtained by application of (2). This result would appear obvious considering that for deterministic fractals, combining relations (13) and (22) on the basis of the assumption that $Z/M = L/\Delta = \eta$ produces relation (2). However, the above assumption does not hold for the river networks we have analyzed. In our sample the ratio L/Δ is generally greater than Z/M for larger basins and the contrary occurs for the smaller ones. This inequality has some impact in the actual estimations, since applying relation (22) with $\eta = Z/M$ on the whole set of basins we obtained $\bar{D}_s = 1.090 \pm 0.093$, with a notable increase of the standard deviation with respect to the results of Table 4.

In conclusion, results obtained in this analysis cannot be considered by themselves a proof of validity of rule (2) for natural basins. However, the results for estimation of D_t and D_s are well supported by literature data and, even though the same support doesn't apply to the estimates of D, these latter share with the former the very small amount of variability found on the set of basins considered. This homogeneity substantiates the validity of (2) with regard to the average values, and suggests that at least at a regional scale, basins tend to the same geomorphologic patterns.

3.4. Variants of the Estimation Method

Fractal dimensions estimated through relations (11), (13), and (22) are the result of an average involving all of the subbasins within the network. Therefore this estimation gives considerable weight to the small subbasins, whose number increases with decreasing M and Δ. One can avoid this effect by grouping the subbasins according to a ranking parameter (essentially the Horton order and the topological diameter), computing the mean fractal dimension for each group, and using these mean values to compute the final average. To verify if this latter procedure gives rise to significant deviations from the reference values shown in the previous sections, we recomputed correspondingly the fractal dimensions for the whole set of basins.

By grouping the subbasins according to the Horton order, we obtained estimates of the three fractal dimensions that are, in average over all the set of basins, 1.533 ± 0.034, 1.683 ± 0.086, and 1.111 ± 0.018 for $\bar{D}_t$, $\bar{D}$, and $\bar{D}_s$, respectively. Further, dimensions D_t and D_s, whose expressions contain the topological diameter Δ, were also recomputed by ranking the subbasins based on Δ. The average values obtained over the whole sample were $\bar{D}_t = 1.540 \pm 0.057$ and $\bar{D}_s = 1.109 \pm 0.038$. D_t was also estimated as the slope in the linear regression between $E[\ln M]_\Delta$ and $\ln \Delta$ (with intercept set to zero), which again is based on the ranking according to the topological diameter. This last computation produced $\bar{D}_t = 1.546 \pm 0.065$.

From these estimates a slight increase of the regional average of D_t from the value of Table 2 can be recognized. This shift, that in all cases is smaller than one standard deviation, is mainly due to the increase in the values estimated on the larger basins (Ofanto and Candelaro) in which the mean D_t for the grouped subbasins displays an increase with the higher Horton orders.

This lack of homogeneity, which certainly offers an opportunity for further investigation, does not have significant effects on the analysis presented here, which essentially deals with estimation of average or "characteristics" values related to the whole basins. To this end, results of the averages obtained by grouping the subbasins do not add substantial arguments to the discussion on the method to use for estimating fractal dimensions, and we can conclude that relations (11), (13), and (22) can be confirmed as the suggested ones for estimation of D_t, D, and D_s, being the most intuitive and simple to use.

4. Final Remarks

In the ensemble of studies on the scaling properties of river networks, one can seldom find proofs of fractality of rivers or of networks associated with convincing measures of the fractal dimensions. Similarly, the most cited composition rule $D = D_s D_t$, between sinuous and topological fractal dimension, was in our opinion never proven convincingly. Therefore the conjecture that fractal dimensions for elongation and branching of rivers present relatively small departures from some "most likely" values was not substantiated in a satisfying way. On the other hand, in the analysis of river systems it is important to recognize if some large deviations from a "rule" depend on the low efficiency of the estimation tool or on some structural differences between geographical regions.

Moving from these considerations, in this paper new elements are added to the discussion, through defining formulas for the estimation of fractal dimensions obtained using the most basic definitions in fractal geometry and on the basis of the analogy between river networks and deterministic fractal trees. With reference to models of deterministic fractal trees, composition of the branching and sinuosity mechanisms has also been reconsidered, so that the rule $D = D_s D_t$ is supported with arguments that also clarify some previous contribution to this topic.

Relations (11), (13), and (22) are suggested for estimation of fractal dimensions D_t, D, and D_s, respectively, as mean values computed throughout all subnetworks in the basin. The results obtained on a sample of 23 basins in southern Italy, in addition to other data arising from the literature, allow us to draw the following conclusions.

1. "Most likely" values of $D_t \approx 1.5$, $D_s \approx 1.1$, and $D \approx 1.7$ arise from this analysis and many literature data; from these results it can be concluded that channel network structures are definitely non-plane-filling.

2. Standard deviations found in our sample related to the estimation of the "most likely" values are below 0.1 for all of the cases, far smaller than comparable deviations coming from applications of previous formulas; the most likely values above, associated with the referred homogeneity, could be taken as a support of the validity of the rule $D = D_s D_t$.

3. Topological fractal dimension D_t is related to parameters of strict topological meaning; the interest in the correct estimation of D_t is related to the findings by *Claps et al.* [1996] about the role of D_t in a maximum-entropy form of the hydrologic response of fractal networks.

4. Elements supporting the use of area-based methods for estimation of fractal dimensions involve some uncertainties and sometimes show a considerable dispersion of results; in our opinion there are no reasons to use areas to estimate parameters such as D_s and D, which are related to "linear" objects, as are river streams. It seems more coherent to use areas to evaluate properties of the "drainage basins," as made, for instance, by *Nikora* [1994].

It is worth pointing out that the investigation of fractal properties of networks was limited here to a "fist-order" analysis, on the basis of the analogy between river networks and deterministic fractal trees, with the goal of highlighting some common network patterns that can be useful in the derivation of the hydrological response. In this approach some points emerged that require further investigations, perhaps in a "second-order" (multifractal or self-affine) analysis. Nevertheless, it is thought that the results collected about the uniformity of patterns among river networks, particularly if supported by further study, can open new perspectives in the study of the hydrological response of river networks.

Appendix 1: Deterministic Fractal Models of River Networks

Construction of deterministic fractal trees is a useful tool as a support to the analysis of self-similar (or self-affine) characteristics of river networks. The visual appearance of synthetic networks whose generation is completely controlled can allow one to discover analogies between some features of fractal trees and of natural networks [e.g., *Claps and Fiorentino*, 1993].

A number of authors have used deterministic models of fractal trees [e.g., *Mandelbrot and Vicsek*, 1989; *Claps and Fiorentino*, 1993; *Nikora and Sapozhnikov*, 1993; *Peckham*, 1995], usually considering limited options in their configurations. The attempt here is to clarify some issues related to models of self-similar trees resulting from composition of models for branching and for sinuosity.

Sinuosity Deriving From Fractality of Individual Segments

A first class of compound models of self-similar trees could be obtained by following the reasoning by *Tarboton et al.* [1990] and *La Barbera and Rosso* [1990], who considered the fractal nature of each single Horton stream in superimposition to the scaling due to branching. In particular, starting from the expression (9) by *La Barbera and Rosso* [1989] for a generic fractal dimension D of river networks, the hypothesis that first-order streams are themselves fractal with dimension d led *Tarboton et al.* [1990] to give for the fractal dimension D, related to the river network as a whole, the expression

$$D = d \cdot \frac{\ln (R_B)}{\ln (R_L)} \tag{A1}$$

and led *La Barbera and Rosso* [1990] to obtain

$$D = \frac{1}{2-d} \cdot \frac{\ln (R_B)}{\ln (R_L)} \tag{A2}$$

Incidentally, application of (A1) and (A2) gives practically the same results for d close to 1.

Arguments against the hypothesis of fractal first-order streams were provided by *Beer and Borgas* [1993], who concluded that on a pure hortonian system, D should be expressed by (9). In practical terms this hypothesis involves that at a given resolution, represented by the length η_Ω of the measuring stick, the euclidean length $l_{1,\Omega}$ is not proportional to η_Ω but to η_Ω^{1-d}, with d as the fractal dimension.

To build a model according to this assumption, one should consider that the growth of the fractal object should occur according to two independent mechanisms: the branching growth, in which links are treated as straight initiators regardless of their internal structure, and the sinuous growth, in which at each replacement the internal structure of links becomes more complex according to the sinuous fractal model.

The condition of independent fractal growth of the network as a topological structure and of the links as individual fractal objects is difficult to achieve in a deterministic fractal tree. The attempts we made produced structures whose constructions weren't compatible with the rules of growth of fractal sets. These rules require that it is clearly stated what is the generator to replace to the euclidean distances between the points that establish the connection rule (that reproduce the initiators), with some possible additional symmetries that uniquely determine the transformation. Even disregarding the problem of drawing sets like these, one should consider that the fractal dimension d of the individual links differs from the fractal dimension D_s of the main stream, which is the one that takes advantage from the collection of empirical results existing in literature.

In conclusion, the hypothesis of fractality of l_1 does not seem suitable for construction of deterministic fractal trees and, considering also the alternative construction discussed below, seems not to be the right one in view of the reproduction of the fractal aspects of the river network as a whole.

As an additional subjective comment, it can be considered that when observing the network at a greater resolution, one should expect to see more details of a previously existing stream, say, between the same two endpoints, and to discover new, small, and simpler, streams. This point of view is reproduced by a model with a fractal main stream and nonfractal first-order streams. On the other hand, if l_1 is assumed fractal, when resolution increases, the new branches appear already as complex as the streams that could be observed previously. In our opinion the former configuration is realistic, while the latter is unrealistic.

Sinuosity Deriving From Fractality of Watercourses

Unlike for individual segments, several clues exist for the fractality of the whole watercourses (see section 3). Composition of this aspect with the self-similarity due to branching can be made as suggested by *Nikora and Sapozhnikov* [1993], who considered a network model obtained by recursive replacement of a generator in which a treelike curve with straight main stream is combined with a Koch curve in order to obtain a fractal mainstream. A typical construction according to this model is shown in Figure 3, for which, according to definition (1), composition of fractal dimensions turns out to be $D = D_sD_t$.

However, this composition rule does not have general validity. Indeed, considering a different topological scheme (Figure 8), with the same Koch curve but a different tree, the appearance of the compound generator can be obtained in different ways, all with the same fractal dimension D, which is no longer equal to D_sD_t.

Considering that the number of links of the mainstream is

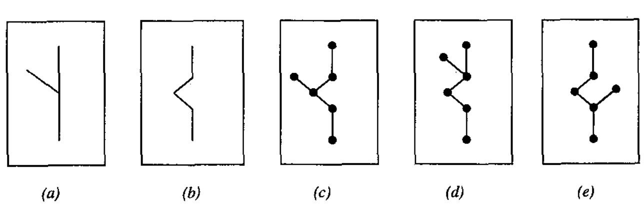


Figure 8. Representation of the possible combinations of two fractal structures. Links between dots represent the initiators for subsequent growth stages. For structure (a) it is $M_t = 3$, $\eta_t = 1/2$, and $D_t = 1.58$; for (b) it is $M_s = 4$, $\eta_s = 1/3$, and $D_s = 1.26$; for structures (c), (d), and (e) $M = 5$, $\eta = 1/3$, and $D = 1.465$.

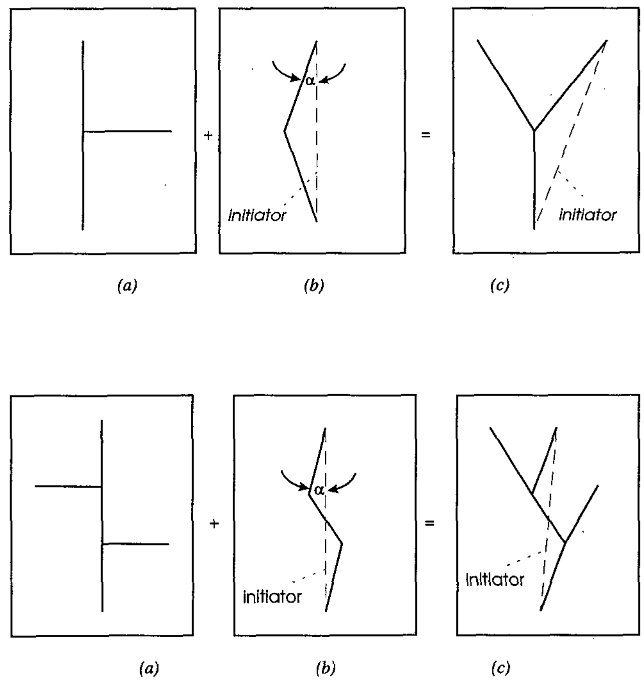


Figure 9. Two examples of construction of fractal structures. Each example shows (a) the topological structure, (b) the sinuous structure, and (c) the resulting fractal structure. The dashed line represents the initiator of the fractal object.

represented at the same time by $1/\eta_t$ (for unit-length initiator) and by M_s, if the condition

$$M_s = 1/\eta_t \tag{A3}$$

is satisfied, the rule expressed by (2) is also respected without exceptions. On the basis of this assumption, equalities $M = M_t$ and $\eta = \eta_s$ produce, for the compound structure, $D = -\ln M/\ln \eta = -\ln M_t/\ln \eta_s = \ln M_t/\ln \eta_t \cdot \ln M_s/\ln \eta_s = D_sD_t$. This condition reduces the options in the drawing of the generator only to considerations of symmetry.

A clean application of the composition of branching and sinuosity requires the initial tree to be a pure topological structure (equal links, squared angles) with unit-length initiator. Figure 9 shows two examples of the building of symmetric fractal trees with sinuous stream paths according to the scheme proposed here. The initiator was also drawn in the figure to clearly define the way these objects grow. It should be noted that the constraint (A3) does not prevent the building of structures with whatever fractal dimensions because a sinuous structure with a given M_s can assume infinite configurations, for example, by varying the angles α between the links (see Figure 9).

Appendix 2: Obtaining $D = D_tD_s$ on Horton Systems

The composition rule (2) can be obtained for hortonian systems without using the hypothesis that l_1 is fractal. To be fractal, a hortonian system needs its properties to be considered asymptotically, as, for instance, the one regarding the dependence of the total network length on the length of first-order streams, expressed by

$$\frac{Z_{\Omega+1}}{Z_\Omega} = \frac{l_{1,\Omega+1}}{l_{1,\Omega}} \cdot R_B \tag{A4}$$

or the dependence of the main stream length on l_1, expressed by

$$\frac{L_{\Omega+1}}{L_\Omega} = \frac{l_{1,\Omega+1}}{l_{1,\Omega}} \cdot R_L. \tag{A5}$$

Note that the latter relation is exact if the length $L_{\Omega,\Omega}$ of the Ωth-order stream is considered. Adopting a ruler length η_Ω proportional to $l_{1,\Omega}$, the increase of Z and of L with the increasing detail due to the change of scale will be governed by

$$\frac{Z_{\Omega+1}}{Z_\Omega} = \left(\frac{l_{1,\Omega+1}}{l_{1,\Omega}}\right)^{1-D} \qquad \frac{L_{\Omega+1}}{L_\Omega} = \left(\frac{l_{1,\Omega+1}}{l_{1,\Omega}}\right)^{1-D_s} \tag{A6}$$

which involve, owing to (A4) and (A5),

$$R_B = \left(\frac{l_{1,\Omega+1}}{l_{1,\Omega}}\right)^{-D} \qquad R_L = \left(\frac{l_{1,\Omega+1}}{l_{1,\Omega}}\right)^{-D_s} \tag{A7}$$

from which one obtains

$$D = \frac{\ln R_B}{\ln R_L} D_s \tag{A8}$$

To verify that the ratio $\ln R_B/\ln R_L$ actually represents the topological fractal dimension, it is easy to recognize that in a model of compound deterministic fractal tree asymptotically results $R_B = M$ and $R_L = M_s$. If the hypothesis (A3) is introduced, it follows also that $M = M_t$, so that

$$\frac{\ln R_B}{\ln R_L} = -\frac{\ln M_t}{\ln \eta_t} = D_t. \tag{A9}$$

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