FRACTAL STRUCTURE, ENTROPY AND ENERGY DISSIPATION IN RIVER NETWORKS

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ABSTRACT. Informational entropy of river networks, as defined by *Fiorentino and Claps* (1992*a*), was shown to be a useful tool to explain several properties exhibited by natural networks. In this paper, self-similar properties of river networks are taken as a starting point for investigating how the entropy of fractal plane trees can be used to show analogies and differences between natural networks and geometric fractal trees. Attention is directed particularly to the relations between entropy and Horton order and entropy and topological diameter of subnetworks. Comparison of features of natural and geometric networks suggest that the use of entropy can contribute to clarify important points concerning regularity properties of river networks in the plane. Furthermore, an interesting comparison is shown in the variability of entropy with the magnitude of subnetworks, for both fractal and natural trees. In natural networks this relation is compared to an index of energy expenditure with the basin size, leading to an intriguing connection between plane and altitudinal features found in river basins.

1. INTRODUCTION

The definition of informational entropy of a river networks can be achieved by considering the network as a system, in which stream segments (*links*) are the elements whose placement characterize the system configuration. A network link is the path connecting two junctions and the *topological distance* from the outlet, *i.e.* the number of consecutive links forming the shortest path from a node to the downstream end, corresponds to the *state* δ in which a link is placed. The total number of states is the network *topological diameter* Δ, corresponding to the maximum topological distance. The network configuration, regardless of links length, is defined by the *topological width function*, which is the diagram of the relative frequency p_δ of the links as a function of the topological distance δ. The informational entropy of river networks, was defined by *Fiorentino and Claps* (1992*a*) as

$$S = -\sum_{\delta=1}^{\Delta} p_\delta \, ln \, p_\delta \qquad (1)$$

consistently with the definition given by *Shannon* (1948) in information theory.

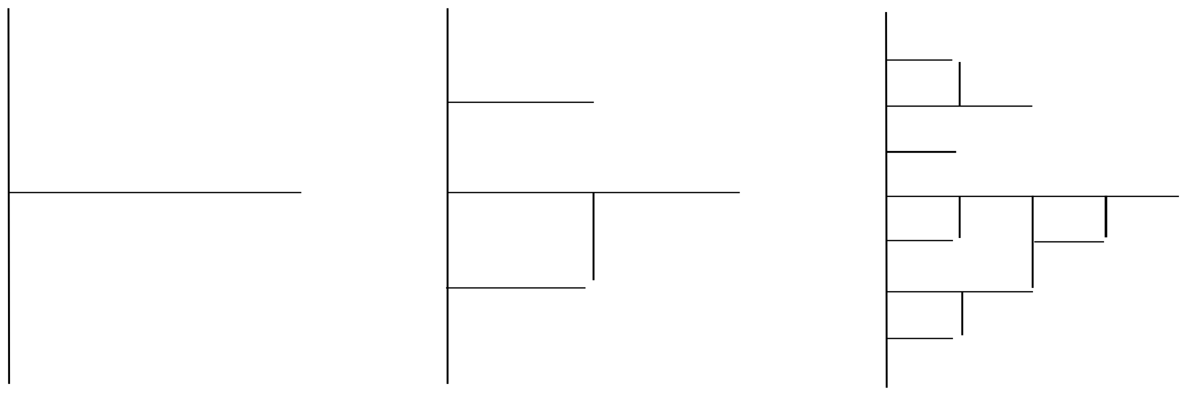
Maximization of the entropy subject to the knowledge of the average elevation of the river network, along with the principle of uniform energy expenditure, led (*Fiorentino et al.*, 1993) to analytical expressions of channel profiles. Also some important scaling properties of channels, such as slope-area and stream power-area relationships, were derived with the use of the fractal dimension D of the branching process of the channel network. Moreover, the informational entropy of river network was empirically found (*Fiorentino et al.*, 1993) to vary with marked regularity with network characteristics, particularly Horton order, magnitude and topological diameter.

In this paper, the above regularities are more deeply investigated using fractal trees, whose properties are first exploited to achieve analytical expressions of the entropy as a function of parameters of their configuration. Entropy properties exhibited by natural river networks are then compared with the properties of the entropy of some geometric fractal trees.

2. INFORMATIONAL ENTROPY OF FRACTAL PLANE TREES

A fractal object can be defined as having a shape made of parts similar to the whole in some way (*Mandelbrot*, 1983, p.34). Fractal plane trees can be built by repeated generations, using an *initiator*, which is a unit-length segment, and a *generator*, which is a tree-type combination of equal shorter segments (*Mandelbrot*, 1983, pp.72-73) whose length is η. After the first substitution of the initiator with the generator, each segment of the generator becomes an initiator and is substituted again, in a recursive way (*e.g. Feder*, 1988, p. 16) as depicted in figure 1. After m generations, the number of segments is $M_m = M_1^m$, where M_1 is the number of segments of the generator, and the segment length is $\zeta_m = \eta^m$. In this paper, generators are taken such that the longest path is straight and that the angles between segments are right. This does not affect generality and allows one to obtain that the number of partitions $1/\eta$ of the initiator equals Δ_1, assumed as the topological diameter of the generator tree. After m generations the topological diameter becomes

$$\Delta_m = \Delta_1^m \qquad (2)$$



m=1 *m=2* *m=3*

Fig. 1. *Generation of a fractal tree. Parameter m is the generation index. The structure with m=1 is the generator. The initiator is a unit-length segment for all the cases considered.*

The fractal dimension D of the so-obtained self-similar geometric set is the exponent describing the fractal (*i.e.* invariant) measure $L=N(\zeta)\,\zeta^D$ of its Euclidean length $L(\zeta)=N(\zeta)\,\zeta$. The latter varies as a function of the unit of measure ζ and of the number $N(\zeta)$ of segments necessary to cover the set. The expression defining D is (*e.g. Feder*, 1988, p.19)

$$D = -\frac{ln\,M_1}{ln\,\eta} \tag{3}$$

When considering river networks, it is to be pointed out that this fractal dimension is only due to the branching process and can be also specified as the *network similarity dimension*, since various others fractal dimensions have been introduced for the description of river networks self-similarities (*e.g. Liu*, 1992; *Beer and Borgas*, 1993).

The topological width function can be readily obtained recursively as a function of *m*, by determining first the number $W_m(j)$ ($j = 1,..,\Delta_m$) of links at the topological distance j from the outlet. It is handy to obtain $W_m(j)$ in Δ_1 sets of length Δ_{m-1}. After the *m*-th generation, given the concept itself of similarity, there will be as many (m–1)-structures as was the number of segments in the generator. The network diameter will then be $\Delta_m = \Delta_1{}^m = \Delta_1\Delta_{m-1}$, leading to Δ_1 sets. In the first set, $W_m(j)$ is not affected by the generation while in the others Δ_1-1 sets the structure of W will reproduce that of the first set multiplied for the number of segments of the generator at the level corresponding to the set number. Operating two generations of a whatever structure clarifies the issue. After *m* generations $W_m(j)$ is obtained as

$$\begin{array}{ll} \textit{set } 1: & W_m(1,...,\Delta_{m-1}) = W_1(1) \cdot W_{m-1}(1,...\Delta_{m-1}) = 1\cdot W_{m-1}(1,...\Delta_{m-1}) \\ \textit{set } 2: & W_m(1\cdot\Delta_{m-1}+1,...,2\cdot\Delta_{m-1}) = W_1(2) \cdot W_{m-1}(1,...\Delta_{m-1}) \\ \dots\dots\dots & \\ \textit{set } \Delta_1: & W_m((\Delta_1-1)\cdot\Delta_{m-1}+1,...,\Delta_1\cdot\Delta_{m-1}) = W_1(\Delta_1) \cdot W_{m-1}(1,...\Delta_{m-1}) \end{array} \tag{4}$$

Informational entropy S of fractal trees, computed using (1) and (4), was shown by *Fiorentino and Claps* (1992*b*) to be: $S_m = S_{m-1} + S_1$. By recursion, this relation produces

$$S_m = m\, S_1 \tag{5}$$

Also, it was shown that this linear relationship does not apply to maximum-entropy and minimum-entropy-production tree structures, which are not self-similar.

In this paper, four kinds of trivalent (*i.e.* three segments joining into each node) generators were used (figure 2) and the properties of the corresponding fractal networks were compared with those of eight natural drainage networks in southern Italy, with characteristics reported in table 1.

3. NETWORK PARAMETERS AND ENTROPY

3.1. Informational entropy and Horton orders

Geometric fractal trees can be analyzed in a hortonian framework, paying attention to the Horton order Ω_1 of the generator tree and to the analogy that M_1 and Δ_1 present with the Horton bifurcation ratio R_B and length ratio R_L, respectively. This analogy is best understood if considering the relation $D = \ln R_B / \ln R_L$ suggested (*La Barbera and Rosso*, 1989) for the network similarity dimension. It will be shown below that this analogy is asymptotically an equality for fractal networks.

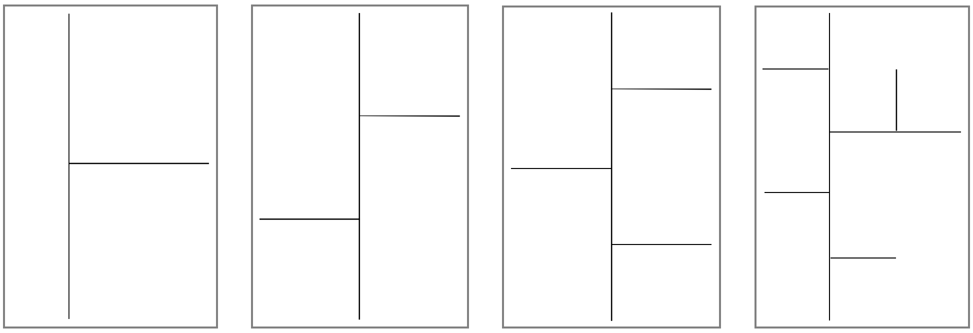

(*a*) (*b*) (*c*) (*d*)

Fig. 2. *Generators of fractal networks:* ***(a)*** M_1=3, η=1/2, D=1.585, S_1=0.637; ***(b)*** M_1=5, η=1/3, D=1.465, S_1=1.055; ***(c)*** M_1=7, η=1/4, D=1.404, S_1=1.352; ***(d)*** M_1=7, η=1/5, D=1.490, S_1=1.516.

Basin	*A (Km²)*	*Dr. Den. (km⁻¹)*	*M. L. (Km)*	*n*	*H (m)*	*E (m)*	Δ	Ω	R_B	R_L
Arcidiaconata	123.9	2.24	23.04	254	657	301	50	5	4.12	2.39
Lapilloso	28.5	2.34	11.56	72	394	229	36	4	4.34	2.28
Vulgano	94.1	2.08	22.08	193	663	370	37	5	3.79	2.26
S.Maria	58.1	2.26	15.53	159	226	144	48	5	3.72	2.57
Salsola	44.1	2.24	14.21	100	513	270	31	5	3.28	2.29
Casanova	57.3	2.20	15.79	123	524	290	26	5	3.44	2.55
Celone S.V.	92.5	2.07	27.59	181	715	362	42	5	3.83	2.73
Celone a P.F.	233.5	1.55	48.61	292	861	405	53	5	4.1	2.74

Tab. 1. *Some characteristics of the Southern Italy basins considered. A (drainage area); Dr.Den. (drainage density); M. L. (mainstream length); n (magnitude); H (total elevation drop of the main stream); E (average basin elevation); D (topological diameter); W (Horton order); R_B (Horton bifurcation ratio); R_L (Horton length ratio).*

Selecting the ruler ζ as the generic link, each generation transforms first-order links into tree structures with higher Horton order, thus producing an increment in the network order. After *m* generations the network order is:

$$\Omega_m = 1 + m(\Omega_1 - 1) \tag{6}$$

Considering (5) and (6), the entropy of subnetworks of order Ω_m is obtained as

$$S_{\Omega_m} = \frac{\Omega_m - 1}{\Omega_1 - 1} \cdot S_1 \tag{7}$$

which is strictly true for structures that can be obtained with a given number *j* of generations (we will call them *j*-structures). For generators of Horton order 2, all subnetworks are *j*-structures, so (7) can be written as

$$S_\Omega = (\Omega - 1)\, S_1 \tag{8}$$

The use of generators of order Ω_1 greater than 2 is awkward, because there is no explicit expressions for the entropy of subnetworks of orders not obtainable through (6). However, the generator in figure 2*d* was considered as an intermediate structure between these of figures 2*a* and 2*b*. For that case, relation (5) is applied only for subnetworks of odd order, which are identical (due to self-similarity) within the network, while entropy of subnetworks of even order is computed as an average value.

The average entropy of natural subnetworks was shown (*Fiorentino et al.*, 1993) to vary linearly with the order, with a slope very close to unity. Since entropy of first-order links is zero, the empirical relation is the same as (8) with S_1 representing the slope of the regression line. S_1 is the rate of increase of entropy both with the index of generation and with the order. It is hence called *rate of entropy production*. Also, S_1 is an estimate of the average entropy of second-order subnetworks.

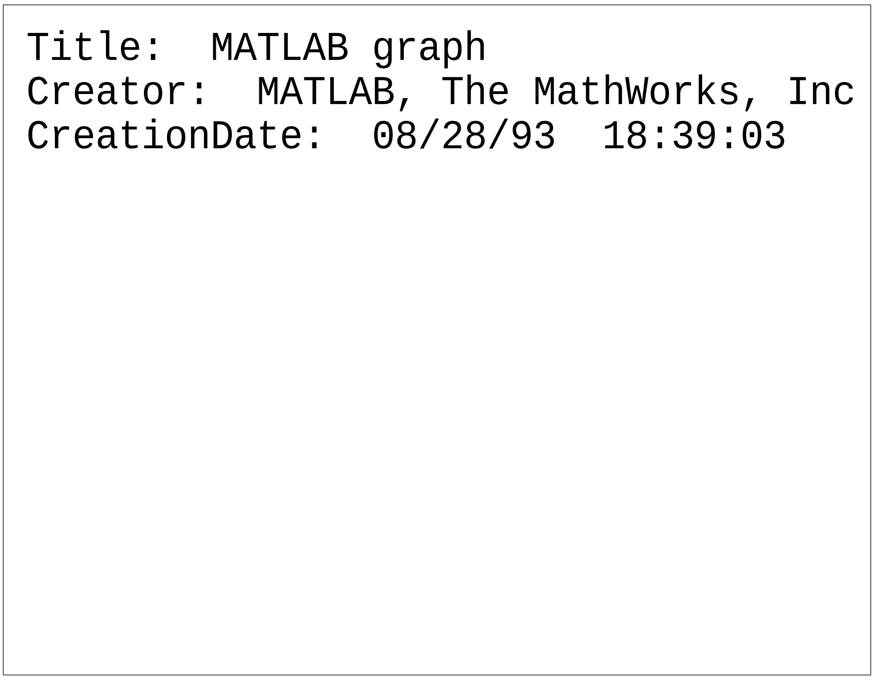


Fig. 3. *Empirical and theoretical linear relations between Horton orders and average entropy of subnetworks.*

In figure 3, theoretical laws obtained for fractal networks are compared with the empirical relations found for the 8 basins considered. It can be noted that natural basins behaves as fractal structures mostly intermediate between these obtained from generators 1*a* and 1*b*. Generator 1*d* could be considered as one of such intermediate structures.

3.2. Informational entropy and topological diameters

Relation (5) can be expressed in terms of the network topological diameter, given (2), as

$$S_{\Delta_m} = \frac{\ln \Delta_m}{\ln \Delta_1} \cdot S_1 \qquad (9)$$

which is again a linear relation between entropy and a network parameter.

It is interesting to comment how close natural basins and fractal networks are to the case of maximum-entropy tree structures, for which entropy is $S_\Delta = \ln \Delta$ (*Fiorentino and Claps,* 1992*a*). To this end, observation of figure 4 reveals that the lines relative to both fractal and natural networks deviate only slightly from the diagonal representing the case of maximum entropy structures. The linear relations between S_Δ and $\ln \Delta$ for the river networks considered were found by regression, with very high correlation coefficients (> 0.99). The dotted lines representing these relations are comprised between the lines of the two fractals obtained from 1*a* and 1*b*.

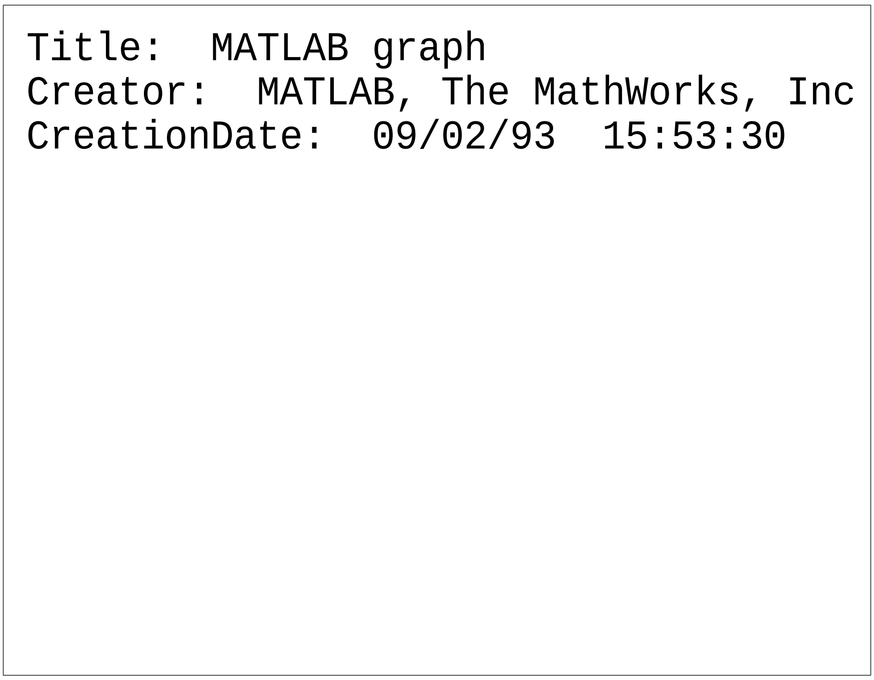


Fig. 4. *Empirical and theoretical linear relations between average entropy of subnetworks with diameter D and ln D.*

In figure 4 is also shown the curve relative to the Peano fractal network (*e.g. Marani et al.*, 1991) which presents a clearly different behavior. The Peano network is a tetravalent structure (four links join into each node) with fractal dimension 2. Thus it is a plane filling curve. In addition to the comments by *Rodriguez-Iturbe et al.* (1992), *Fiorentino and Claps* (1992*b)* and *Rinaldo et al.* (1992), the position of the line relative to the Peano network in figure 4 is a further confirmation that this tetravalent fractal tree is not representative of the features of natural networks.

3.3. Informational entropy and fractal dimension

The analogies in patterns exhibited by fractal and natural networks with reference to their informational entropy suggest to reconsider the usual estimation of the network similarity dimension proposed by *La Barbera and Rosso* (1989). In this regard, *Beer and Borgas* (1993) recently highlighted on a natural network the sensitivity of the relation $D = ln\ R_B / ln\ R_L$ with the basin order. We have derived theoretically a similar conclusions for fractal networks.

Letting fractal trees grow by successive generations we have computed the Horton ratios R_B and R_L to obtain an estimate of $D = ln\ R_B / ln\ R_L$ for each order of generation. As shown by figures 5 and 6, relative to the networks obtained by generators *a* and *d* of figure 2, this estimate of D is correct only asymptotically for purely fractal networks and does not depend on the order Ω_1 of the generator (*Claps and Oliveto*, 1993). Based on this result and on the number of linear relations found between entropy and fractal network properties, a direct relation between S and D could be used in order to improve the estimate of D.

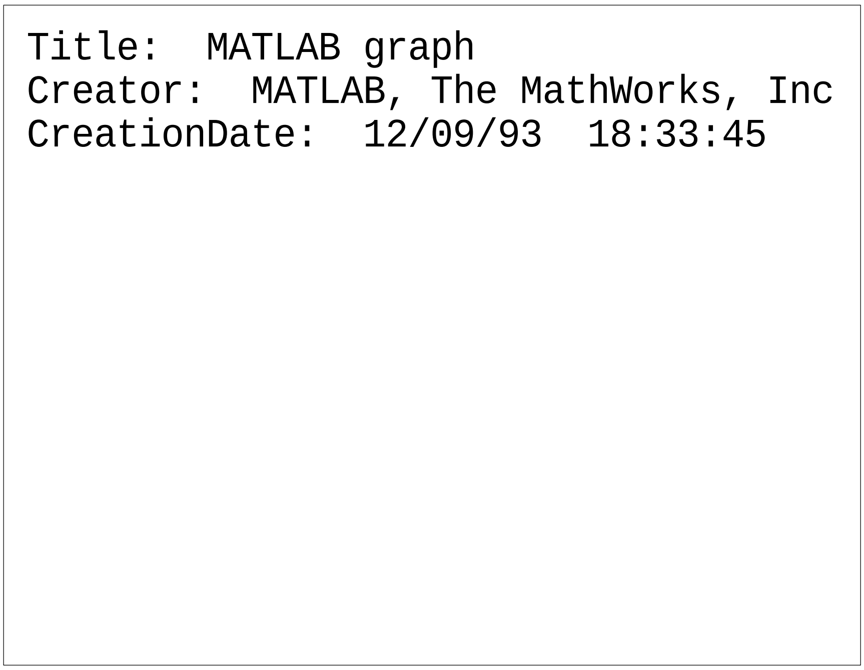


Fig. 5. *Estimate of D as ln R_B / ln R_L obtained for different levels of generations of the fractal network derived from the generator in figure 2a.*

In fact, a relation between D and S_1 is apparent for geometric fractal trees. Using data from a number of fractal structures with generators of second order, including those of figure 2, we have obtained the regression relation

$$D = 1.476 - 0.240\, ln\, S_1 \qquad (10)$$

between the two variables (figure 7). This relation can be used to estimate D once S_1 is known.

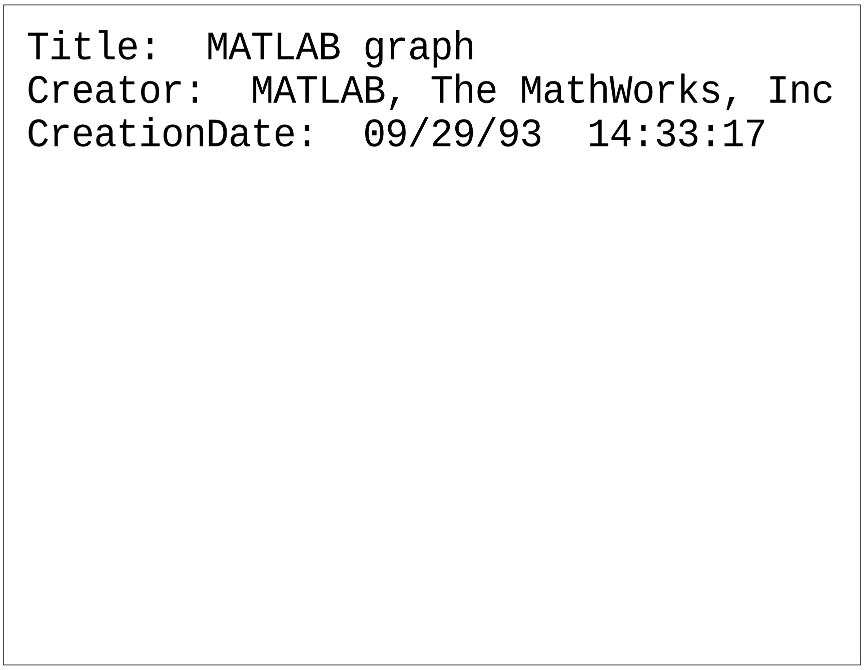


Fig. 6. *Estimate of D as ln R_B / ln R_L obtained for different levels of generations of the fractal network derived from the generator in figure 2d.*

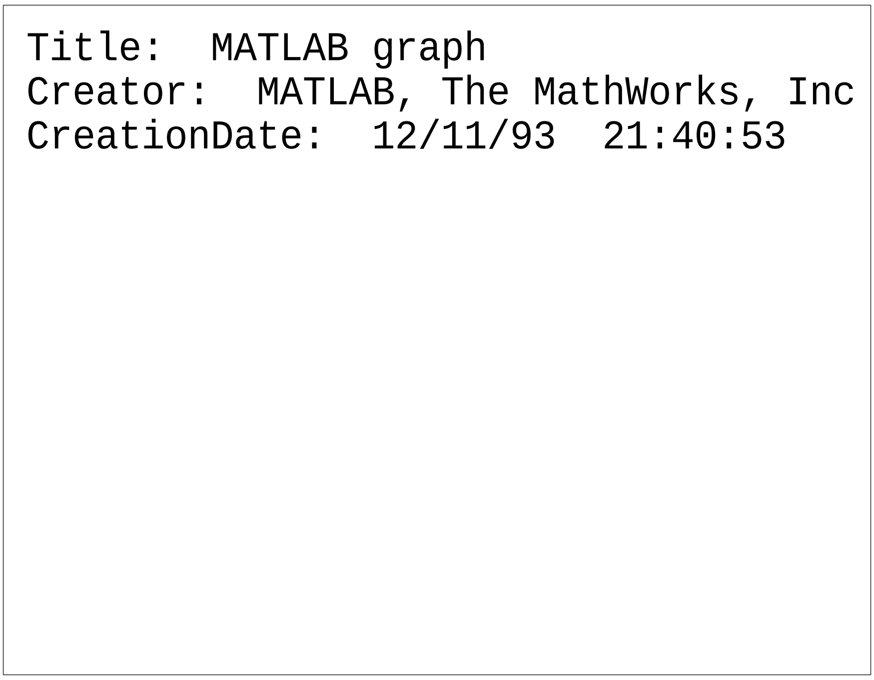


Fig. 7. *Regression between the logarithm of the theoretical fractal dimension D and the logarithm of the rate of entropy production* S_1 *for fractal networks with second-order generators.*

For the eight natural basins considered, application of (10) produced very interesting results, reported in table 2. For these networks S_1 was evaluated by the regression between the average entropy of subnetworks of each Horton order and the order itself (relation (8)). For natural networks, in comparison to the estimates of $ln\ R_B$ / $ln\ R_L$, values of D obtained through (10) are much more stable around the mean (see table 2), possibly denoting the robustness of the estimation method.

Basin	lnR_B/lnR_L	S_1	D
Lapilloso	1.78	1.081	1.43
Celone F.S.	1.40	0.981	1.47
Arcidiaconata	1.63	0.952	1.48
Celone S.V.	1.34	0.953	1.49
Vulgano	1.64	0.880	1.49
Salsola	1.43	0.837	1.51
Casanova	1.32	0.859	1.53
S.Maria	1.39	1.018	1.49
Average	*1.49*		*1.49*
Std. Dev.	*0.169*		*0.029*

Tab. 2. *Estimates of* $ln\ R_B / ln\ R_L$ *and of D (using relation 10) for natural networks of Table 1. Values of* S_1 *needed in relation 10 are estimated by regression with the order (relation 8).*

4. ENTROPY AND POTENTIAL ENERGY

The potential energy of a basin is given by the elevation of the network nodes above a datum (say, the basin outlet). For establishing a relation between entropy and potential energy, we consider y_δ as the mean node elevation at the topological distance δ from the outlet. We take $\overline{y}$ as the mean elevation for all the nodes, averaging y_δ with δ varying from 1 to Δ, and take it as approximately the mean basin elevation. $\overline{y}$ is the total potential energy of the drainage-network system.

Fiorentino *et al.* (1993) suggested a link between mean basin elevation and entropy:

$$\overline{y} = -\alpha \ln\beta + \alpha S \tag{15}$$

holding for any drainage subnetwork within the larger basin. This relation was based on the assumption that for the drainage-network system, the distribution of potential energy can be assumed to be controlled by the two fundamental quantities Δ (topological diameter) and T (representing a degenerate temperature of the drainage network) both measured for the entire system. Based on the analogy with a thermodynamic system where T can be thought of as proportional to the energy content of the system, α and β may be assumed to be constant as a first-order approximation.

In this paper, we advocate some properties found for fractal networks in order to further clarify the role of entropy, determined on the basis of 2-dimensional information, in the interpretation of the potential energy distribution within the basin. This investigation tends to put together all information related to the state of natural networks in the hypothesis that they adjust their geometry to achieve an optimal configuration in terms of minimum global energy expenditure of the runoff they convey.

The starting point of the comparison between natural and fractal networks must be something connecting the spatial to the plane network structures. The comparison of figures 8 and 9 provides interesting elements in this regard. Figure 8 shows for two basins, taken from the group of eight, the variability of the average elevations of a series of subnetworks against their entropy. For the sake of simplicity, we have considered only the subnetworks directly draining into the main stream. In this way, the average increase of mean elevation relative to the outlet reflects the increase in subnetwork size (and in its relative relief, or elevation drop) moving from upstream to downstream. Figure 9 represents the relation between logarithm of magnitude and entropy for the same set of subnetworks of the two basins.

As is apparent from the figures, sudden increases both in magnitude and in mean elevation occur with a minimum (sometimes almost null) increase in entropy. Discontinuities in both kind of curves occur at the same points, denoting the existence of important junctions which considerably increase the magnitude and increase or decrease the mean elevation. In the regions between discontinuities both magnitude and mean elevation display uniform variability with entropy, represented by straight lines with almost constant slope.

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Fig. 8. *Average elevation* $\overline{y}_d$ *versus informational entropy* S_d *of subnetworks whose outlets lie on the main channel, for (A) Salsola and (B) Vulgano basins.*

One of the interesting aspects in comparing this behavior is the fact that not only the distribution of potential energy in subnetworks draining into a path varies not uniformly with the "complexity" of the topological structure (represented by entropy) but also disuniformities are concentrated in few important junctions, identifiable when a great increase occurs in network magnitude.

In the sub-paths between the significant junctions mean elevation decreases gradually with network complexity while magnitude obviously increases. The decrease of mean elevation can be easily understood considering that first-order or second-order streams joining streams of greater order are placed in the downstream end of the subnetwork, so their average elevations are generally below the mean network elevation.

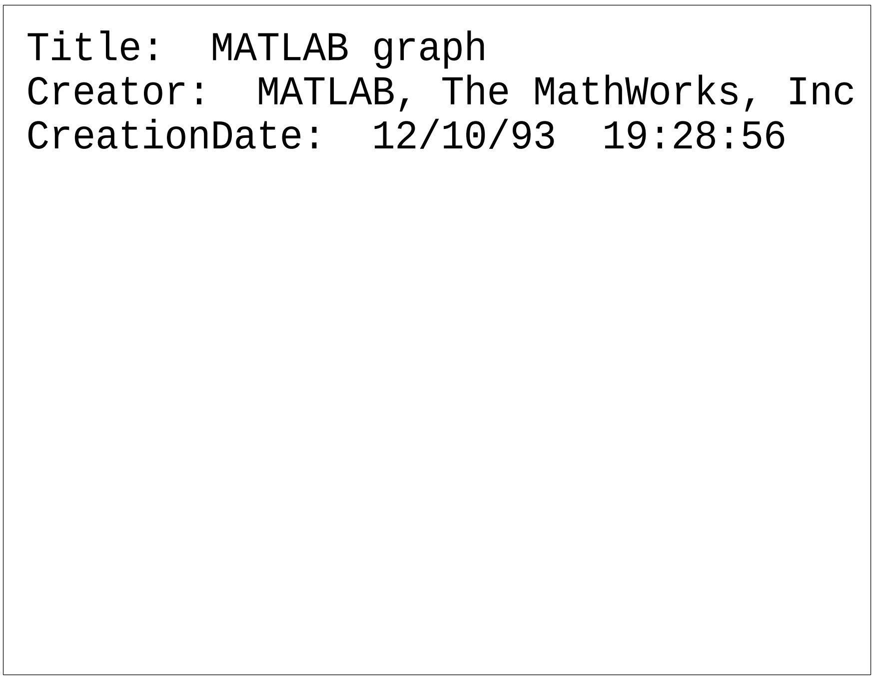

Fig. 9. *Magnitude versus informational entropy of subnetworks whose outlets lie on the main channel, for (A) Salsola and (B) Vulgano basins.*

The parallelism that has been established between the variability of magnitude and of mean-elevation with entropy justifies the interest in investigating the structure of the variability of subnetwork magnitude with entropy in a fractal framework. To this end, comparison of the patterns shown in figure 10 for a fractal network with these observable in figure 9 for natural networks clarifies that, even with regard to magnitude, entropy varies in natural basins as if they were fractal objects. Figure 10 is built plotting the average entropy of all subtrees of the fractal network versus their magnitude. All of the subnetworks drain into one of the several equal main channels of the network, so the average entropy is computed among identical subnetworks and this representation is equivalent to that of the two previous figures.

For a fractal tree, the variability of entropy with magnitude can be characterized completely. Let us first make reference to *m*-structures, for which (by construction) the total number of segments increases with *m* as

$$\mathrm{M}_m = \mathrm{M}_1^m \tag{11}$$

The above relation allows us to substitute *m* into (5) to obtain

$$\mathrm{S}_m = \frac{\mathrm{S}_1}{ln\,\mathrm{M}_1} \cdot ln\,\mathrm{M}_m \tag{12}$$

Considering that the magnitude *n* is equal to (M+1)/2, for large values of *n* one can substitute $n_m = \mathrm{M}_m/2$ into (12) to have the desired relation between entropy and magnitude:

$$\mathrm{S}_m = \frac{\mathrm{S}_1}{ln\,\mathrm{M}_1} \cdot ln\,2n \tag{13}$$

In figure 10 it can be easily recognized that the points relative to *j*-structures are the ones preceding the steps with high increase in magnitude and low increase in entropy. The line representing relation (13) connects all these points while intermediate structures behaves differently. It is worth noting that in fractal networks the step in magnitude corresponds to an increase in Horton order.

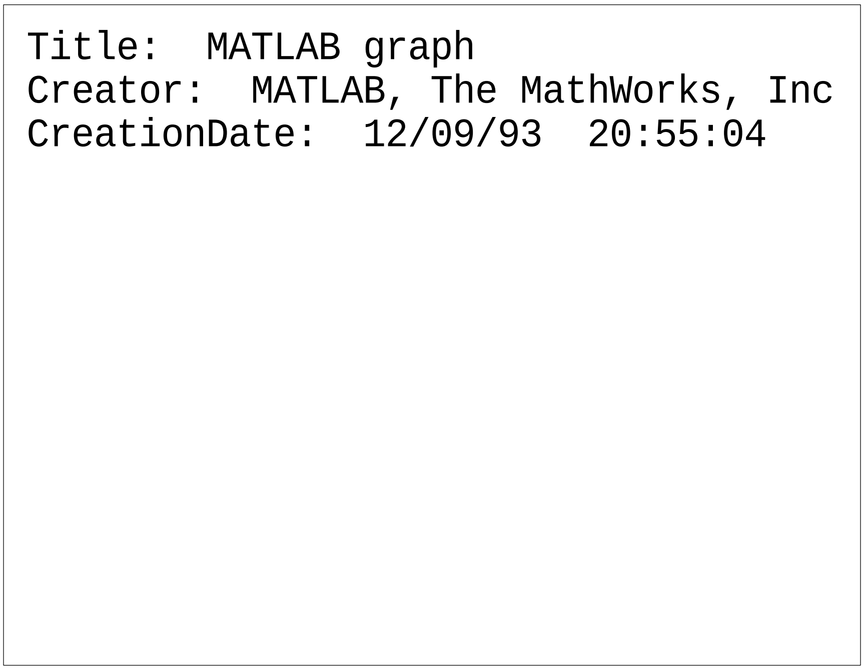

Fig. 10. *Magnitude versus average informational entropy of subnetworks for the network obtained after 5 generations of the structure in fig. 1a. Magnitudes of the j-structures (j=1,..,5) are 2, 5,14,41,122.*

Intermediate structures present a greater increase of entropy with magnitude than that occurring for *j*-structures and this different slope becomes stable for relatively large networks. To explain this it is convenient to make reference to the way the diagram of figure 10 is built. Following one of the several equal main paths one finds all possible structures existing in the network. Each step downstream increases of one unity the topological diameter of the subnetwork considered. For *m*-structures, topological diameter and total number of links are related by

$$M_m = \Delta_m^D \tag{14}$$

which is obtained by elimination of *m* between (2) and (11) and considering that in (3) $\eta = 1/\Delta_1$. So, for the points (S,*n*) relative to *m*-structures relation (13) gives

$$S_m = \frac{S_1}{ln\, M_1} \cdot D\, ln\, \Delta_m \tag{15}$$

that means entropy S_m is exactly proportional to the *primary entropy* $ln\Delta_m$ (maximum entropy for structures of diameter Δ_m). But we have seen in section 3.2 that the whole fractal network is a quasi maximum-entropy structure, meaning that $S = C\, ln\Delta$, with $C \approx 1$.

Thus the difference between the *m*-structures and all the intermediate structures is in the approximation achievable with (16), because by increasing Δ entropy always increases approximately as $ln\Delta$ while magnitude increases as n_m=1/2 Δ^D for the *m*-structures and much less for intermediate structures.

5. FINAL REMARKS

Some important properties displayed by parameters of natural basins connected to the concept of informational entropy of the network are explored in this paper. The fractal nature of river network allowed us to make use of purely fractal trees to further substantiate self-similarity properties in natural basins and to better understand patterns of variability of some parameters as functions of the informational entropy. To this end, exact relations between entropy and fractal network parameters are derived in many cases.

The following points are worth emphasizing:

1) the entropy of fractal and natural networks is linearly related, in average, to the Horton order, with the slope of the relation, representing the rate of entropy production, strictly related to the fractal dimension of the branching process;

2) fractal and natural networks behaves as quasi maximum-entropy structures, allowing us to use the logarithm of the topological diameter as a good approximation for the network entropy;

3) the connection between rate of entropy production and the similarity dimension of fractal network permits an estimation of the fractal dimension of the branching more stable than the usual estimate $D=ln\ R_B/ln\ R_L$, which is shown only asymptotically correct with regard to fractal trees;

4) the distribution of potential energy and of magnitude along the main stream of natural networks is shown to vary with similar patterns with respect to the entropy of the subnetworks. Both variables display discontinuities in correspondence of significant junctions along the path and gradual variations between these junctions. With regard to magnitude versus entropy again fractal and natural networks show the same patterns, which are given a satisfying interpretation with the aid of the rules valid for fractal trees.

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